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The Bethe Ansatz for the Six-Vertex Model with Rotated Boundary Conditions

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A method is suggested for derivation of the Bethe ansatz equations for the six-vertex model on a square lattice rotated at an arbitrary angle with respect to the coordinate axes. The method is based on the random walk representation for configurations of the model. The equations for the ice model on the rotated lattice are derived and some numerical results are obtained.

KEY WORDS: Bethe ansatz; six-vertex model; random walk representation; rotated boundary conditions.

1. INTRODUCTION

In this paper we develop some methods of solution of the six-vertex model with various boundary conditions. Our aim is to apply the random-walk formalism to derive the Bethe ansatz equations for a lattice rotated through an arbitrary angle with respect to its "natural" orientation.

The original approach by Lieb[1] used the transfer matrix method for a square lattice wrapped in a torus along rows and columns of the lattice. Temperley and Lieb[2] considered the ice rule models on a square lattice drawn diagonally and found that this geometry is more convenient for comparison of the one-dimensional quantum and two-dimensional vertex models.[3] Pegg[4] used techniques outlined in ref. 3 to show how the six-vertex model can be solved with a diagonal-to-diagonal transfer matrix.

Whereas the Bethe ansatz equations for these basic orientations have been well investigated, almost nothing is known for other orientations. The rotation of a lattice induces some complication of boundary conditions. These are considered for the following reasons.

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Let us draw a rotated square lattice on a strip of width M and consider an ice-type model with two types of boundaries: open edges and glued ones which transform the strip into a cylinder. Each bond can accept one of two states characterized by arrows. Besides the ice rule for inner sites, one usually assumes reflecting conditions for the boundary sites in the case of open edges: at each boundary site there is one arrow in and one arrow out.

Instead of arrow configurations, one often considers a line representation, drawing a line on a bond if a corresponding arrow points down, otherwise leaving the bond empty. Typical arrangements of lines for the angle of rotation $\alpha = \pi/4$ are shown in Fig. 1. A comparison between Figs. 1a and 1b shows that reflecting conditions force the lines to have an average orientation along the strip which reduces the entropy if the angle of rotation differs from $\pi/4$. In the limit $M \to \infty$ for periodic conditions we expect the free energy f_M to tend to a known value f_∞ independent of the angle of rotation α. On the contrary, in the case of an open strip with reflecting conditions one should obtain a limit value $f(\alpha)$ which varies with α from f_∞ at $\alpha = \pi/4$ to its trivial value at $\alpha = 0$ when all lines becomes straight and parallel to the vertical axis.

Although the qualitative behavior of the free energy is clear, investigation of its analytic form needs the solution of the Bethe ansatz equations for all α. Note that the situation here is drastically different from those in Ising-type models, where the equivalence of open and periodic boundary conditions as well as various periodic conditions in the thermodynamic limit follows from the Van Hove theorem. The difficulties in proving the equivalence of various boundary conditions in the case of the six-vertex model are discussed in the thorough review in ref. 3.

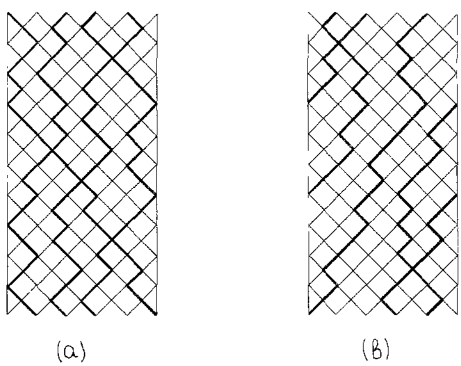

Fig. 1. Typical arrangements of lines representing the ice model on the strip: (a) reflection boundary condition; (b) periodic boundary condition.

More important problems arise when considering finite-size effects in this model. It is known[5] that for a conformal-invariant model with periodic boundary conditions the free energy per site is

$$f_M = f_\infty - \frac{1}{M^2}\frac{\pi}{6}C + \text{higher order terms} \tag{1}$$

where C is the central charge of the Virasoro algebra. De Vega and Karowski[6] used Eq. (1) to obtain C from $1/M^2$ corrections in the case of the normal orientation of the lattice. For a lattice of finite width M, the angular dependence of f_M to be expected even for periodic conditions calls for an investigation of correction terms in the case of arbitrary orientations. The proof of the independence of $1/M^2$ corrections of the lattice orientation would be an argument for the rotation invariance of the six-vertex model, which is not yet proved by a direct calculation of correlation functions.

Also, if one introduces the transfer matrix along the strip with $\exp(-a\hat{H})$, where a is a properly chosen lattice spacing, $\hat{H}$ may be thought of as the Hamiltonian operator of a quantum field theory in $(1+1)$ dimension. So, the six-vertex model on rotated lattices generates a variety of $(1+1)$ field theories parametrized by the angle of rotation.

In this paper we are concerned with the six-vertex model solely for periodic boundary conditions. We will use the line representation, considering each arrow configuration as a set of random walks restricted by the ice rule. Our aim is to express the number of these configurations by generating functions of simple unrestricted random walks. It appears that this program is completely equivalent to the Bethe ansatz approach, with the exception of using a transfer matrix which may be of a very complicated form for an arbitrary angle of rotation.

In Section 2 we formulate the problem and consider a single random walk on a square lattice embedded on the surface of a cylinder and rotated with respect to the cylindrical axis. In Section 3 the Bethe ansatz equations for this model are derived. In Section 4 the known results are obtained from these equations. In Section 5 some numerical results for the ice model on the rotated lattice are presented.

2. SINGLE RANDOM WALK ON ROTATED LATTICE

Let us start by considering an auxiliary square lattice $\mathscr{L} = \mathscr{L}_x \times \mathscr{L}_y$ wrapped on a cylinder along the columns of $\mathscr{L}$. Let us draw on $\mathscr{L}$ the basic lattice L with the translation vectors (a_1, a_2) and $(a_2, -a_1)$, where

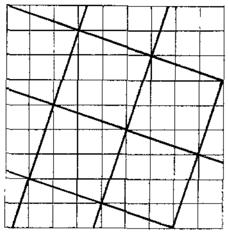

Fig. 2. The auxiliary lattice $\mathscr{L}$ (thin lines) and the basic lattice L (bold lines) with the translation vectors (1, 3) and (3, −1).

a_1, a_2 are integers and $a_2 > a_1$ for definiteness. In Fig. 2 the case $a_1 = 1$, $a_2 = 3$ is shown.

To get periodic boundary conditions on L, we require the number of columns $\mathscr{L}_x$ to be divisible by $(a_1^2 + a_2^2)$. Each bond of L is provided by an arrow so as to obey the ice rule: there are always two arrows pointing away and two pointing into each site of L. The six possible states of vertices are shown in Fig. 3. Let $\xi(i) = 1, 2,..., 6$ be an index of the vertex configuration at the ith site of L and $\omega_\xi \equiv \exp(-\beta\varepsilon_\xi)$ be the Boltzmann weight of a vertex having energy ε_ξ; β is the inverse temperature. The problem consists in determining the partition function

$$Z = \sum_G \sum_{i \in L} \omega_{\xi(i)} \tag{2}$$

where the sum runs over all possible arrow configurations G on the lattice L.

To formulate the problem in terms of restricted random walks, we identify an arrow pointing down with a segment of a walk. The configuration of the model is represented by n walks oriented along the cylinder axis and not intersecting each other. Due to the ice rule, two walks are able to have sites of contact on L, but are not bonds. The ice rule provides also

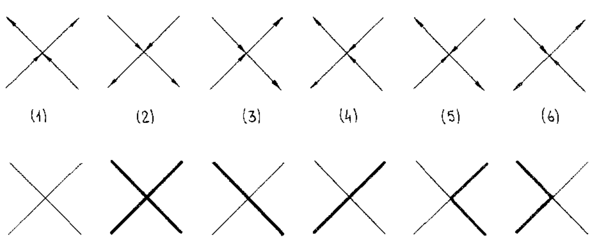

Fig. 3. The ice-rule arrow configurations and the line representation.

the conservation of the number of walks n at each cross section of the cylinder.

As usual, we begin an enumeration of possible configurations of the model with the case $n=1$. This case corresponds obviously to a single unrestricted random walk on the lattice L. It is convenient, however, to describe this random walk in terms of the auxiliary lattice $\mathscr{L}$. Consider an arbitrary walk starting from an upper row of $\mathscr{L}$ and ending at a lower one. We identify the number of a row with discrete time t $(0 \leqslant t \leqslant \mathscr{L}_y)$. Denote the coordinate of a site of L on the upper row of $\mathscr{L}$ by x_0.

Let $w_t^l(x)$ and $w_t^r(x)$ be weighted sums of walks starting at x_0 and terminating at x at a moment t from the left and from the right, respectively. The weight ascribed to each walk is determined by the weights of the line configurations at all sites passed by the walk. Bearing in mind that the walks go along bonds of L and each site of L not passed by the walk has the weight ω_1, we write the recurrence equations

$$\begin{aligned} w_t^l(x) &= \frac{\omega_3}{\omega_1} w_{t-a_1}^l(x-a_2) + \frac{\omega_5}{\omega_1} w_{t-a_1}^r(x-a_2) \\ w_t^r(x) &= \frac{\omega_6}{\omega_1} w_{t-a_2}^l(x+a_1) + \frac{\omega_4}{\omega_1} w_{t-a_2}^r(x+a_1) \end{aligned} \tag{3}$$

We are interested only in random walks on the lattice L; therefore we may put $w_t^r(x) = w_t^l(x) = 0$ for all sites $(x, t) \in \mathscr{L}$ not coinciding with sites of L. Defining the generating functions

$$w^\alpha(x) = \sum_{t=0}^{\infty} w_t^\alpha(x) z^t, \qquad \alpha = l, r \tag{4}$$

we obtain from Eqs. (3) and (4)

$$\begin{aligned} w^l(x) &= \frac{\omega_3}{\omega_1} z^{a_1} w^l(x-a_2) + \frac{\omega_5}{\omega_1} z^{a_1} w^r(x-a_2) \\ w^r(x) &= \frac{\omega_6}{\omega_1} z^{a_2} w^l(x+a_1) + \frac{\omega_4}{\omega_1} z^{a_2} w^r(x+a_1) \end{aligned} \tag{5}$$

It is necessary to distinguish two solutions of Eq. (5) corresponding to different initial conditions:

(a) The first step is left: $w_0^l(x) = \delta_{xx_0}$, $w_0^r(x) = 0$; denote the solution by $\tilde{w}^\alpha(x)$.

(b) The first step is right: $w_0^r(x) = \delta_{xx_0}$, $w_0^l(x) = 0$; denote the solution by $\tilde{\tilde{w}}^\alpha(x)$.

A discrete Fourier transformation

$$w^{\alpha}(k)=\sum_{x=1}^{\mathscr{L}_x} w^{\alpha}(x)e^{ikx}, \qquad \alpha=l, r \tag{6}$$

yields

$$\begin{pmatrix} 1-\dfrac{\omega_3}{\omega_1} z^{\alpha_1} e^{ika_2} & -\dfrac{\omega_5}{\omega_1} z^{a_1} e^{ika_2} \\ -\dfrac{\omega_6}{\omega_1} z^{\alpha_2} e^{-ika_1} & 1-\dfrac{\omega_4}{\omega_1} z^{a_2} e^{-ika_1} \end{pmatrix} \begin{pmatrix} w^l(k) \\ w^r(k) \end{pmatrix} = I_0 \tag{7}$$

where

$$I_0=\begin{pmatrix} e^{ikx_0} \\ 0 \end{pmatrix}$$

for initial conditions (a) and

$$I_0=\begin{pmatrix} 0 \\ e^{ikx_0} \end{pmatrix}$$

for (b). Defining

$$\Delta=\det\begin{pmatrix} 1-\dfrac{\omega_3}{\omega_1} z^{a_1} e^{ika_2} & -\dfrac{\omega_5}{\omega_1} z^{a_1} e^{ika_2} \\ -\dfrac{\omega_6}{\omega_1} z^{a_2} e^{-ika_1} & 1-\dfrac{\omega_4}{\omega_1} z^{a_2} e^{-ika_1} \end{pmatrix} \tag{8}$$

we obtain the solutions of Eq. (7)

$$\begin{aligned} \tilde{w}^l(k) &= \Delta^{-1} e^{ikx_0}\left(1-\frac{\omega_4}{\omega_1} z^{a_2} e^{-ika_1}\right) \\ \tilde{w}^r(k) &= \Delta^{-1} e^{ikx_0} \frac{\omega_6}{\omega_1} z^{a_2} e^{-ika_1} \end{aligned} \tag{9a}$$

and

$$\begin{aligned} \tilde{\tilde{w}}^l(k) &= \Delta^{-1} e^{ikx_0} \frac{\omega_5}{\omega_1} z^{a_1} e^{ika_2} \\ \tilde{\tilde{w}}^r(k) &= \Delta^{-1} e^{ikx_0}\left(1-\frac{\omega_3}{\omega_1} z^{a_1} e^{ika_2}\right) \end{aligned} \tag{9b}$$

On the basis of definitions (4) and (6) we have

$$w_t^\alpha(k)=\frac{1}{2\pi i}\oint\frac{dz}{z^{t+1}}\,w^\alpha(k), \qquad \alpha=l,r \tag{10}$$

The path of integration in Eq. (10) is a circle of radius $r<|z_0(k)|$, where $z_0(k)$ is a root minimal in absolute value of the equation

$$\Delta=0 \tag{11}$$

Using Eqs. (9a), (9b), and (10) and introducing $\lambda_k=z^{-1}(k)$, we obtain

$$\begin{aligned}\tilde{w}_t(k)&=\tilde{w}_t^l(k)+\tilde{w}_t^r(k)\\ &=e^{ikx_0}\lambda_k^{t+1-a_2}\left(\lambda_k^{a_2}-\frac{\omega_4}{\omega_1}e^{-ika_1}+\frac{\omega_6}{\omega_1}e^{-ika_1}\right)\end{aligned} \tag{12a}$$

and

$$\begin{aligned}\tilde{\tilde{w}}_t(k)&=\tilde{\tilde{w}}_t^l(k)+\tilde{\tilde{w}}_t^r(k)\\ &=e^{ikx_0}\lambda_k^{t+1-a_1}\left(\lambda_k^{a_1}-\frac{\omega_3}{\omega_1}e^{ika_2}+\frac{\omega_5}{\omega_1}e^{ika_2}\right)\end{aligned} \tag{12b}$$

Here λ_k can be interpreted as an eigenvalue of a transfer matix T coupling two states of adjacent rows of $\mathscr{L}$ with an eigenfunction e^{ikx}. The action of the matrix T on a coordinate function of type e^{ikx} can be described as follows.

Consider a walker at a site of L. The matrix T moves it in one or another direction along the bond of L down to crossing the next row of $\mathscr{L}$. The movement continues until reaching the next site of L. Then the coordinate function is multiplied by e^{-ika_1} or e^{ika_2} and by ω_ξ/ω_1, depending on the line configuration at the given site.

We do not use, however, the explicit form of the transfer matrix T, and later it is important merely that $w_t(k)$ is actually the generating function of all unrestricted t-step walks with the weights e^{-ika_1} and e^{ika_2} ascribed to the left and to the right step along the bonds of the lattice L and with the weights ω_ξ/ω_1 ascribed to each site of L passed by a walk.

From Eqs. (12a) and (12b) we get the leading asymptotics of $\tilde{w}_t(k)$ and $\tilde{\tilde{w}}_t(k)$ as $t\to\infty$:

$$\tilde{w}_t(k)\sim\tilde{\tilde{w}}_t(k)\sim\lambda_k^t \tag{13}$$

We also define a "phase" factor $\phi(k)$:

$$\tilde{w}_{t-a_1}(k)=\phi(k)\,\tilde{\tilde{w}}_{t-a_2}(k) \tag{14}$$

which due to Eqs. (12a) and (12b) has the form

$$\phi(k) = \frac{\lambda_k^{a_2} + e^{-ika_1}(\omega_6 - \omega_4)/\omega_1}{\lambda_k^{a_1} + e^{ika_2}(\omega_5 - \omega_3)/\omega_1} \tag{15}$$

3. THE BETHE ANSATZ

We consider now n walks described by the generating functions $w_t(k_1)$, $w_t(k_2)$,..., $w_t(k_n)$ with the "wave numbers" $k_1, k_2,..., k_n$. Following the Bethe ansatz prescription, we define a sum with certain coefficients $a(p)$:

$$Z_t = \sum_p a(p)\, w_t(k_{p(1)})\, w_t(k_{p(2)}) \cdots W_t(k_{p(n)}) \tag{16}$$

over $n!$ permutations of n numbers 1, 2,..., n: $p(1)$, $p(2)$,..., $p(n)$. The goal is to choose a complex-valued function $a(p)$ so that the contributions from all configurations of n walks not satisfying the ice rule cancel out. Then

$$Z_t = \sum_p a(p)\, F_t(k_{p(1)},..., k_{p(n)}) \tag{17}$$

where $F_t(k_{p(1)},..., k_{p(n)})$ is a weighted sum over all configurations of n walks restricted by the ice rule. The walks in F_t are nonintersecting, ordered [i.e., labeled by $p(1)$, $p(2)$,..., $p(n)$] and orientated along the cylinder axis.

Let us now assume that $\{k\}$ is a symmetric set:

$$k_j = -k_{n-j+1}, \qquad j = 1, 2,..., n \tag{18}$$

and that the lattice is wrapped in a torus. Then each walk becomes closed. In this case

$$Z_{\mathscr{L}_y} = F_{\mathscr{L}_y}(0, 0,..., 0) \sum_p a(p) \tag{19}$$

because the total weight of closed walks originating from the factors $\exp(-ika_1)$ and $\exp(ika_2)$ is $\exp[m\mathscr{L}_x(k_1 + \cdots + k_n)]$ where m is the number of synchronous rotations of restricted walks around the vertical axis. Due to (18), this factor equals unity.

The function on the right-hand side of (19) is just what we want to find. By the definition of the partition function (2) we have

$$Z = \omega_1^{|L|} \sum_n F_{\mathscr{L}_y}(0, 0,..., 0) \tag{20}$$

where $|L|$ is the total number of sites of the lattice L and the summation runs over all possible numbers of walks n.

The additional factor $\sum_p a(p)$ in (19) does not play any role in the limit of a very long cylinder or torus because, as we will show below, $a(p)$ is a bounded functions of $k_1,..., k_n$ not depending on $\mathscr{L}_y$, which yields

$$\lim_{\mathscr{L}_y \to \infty} \left[\sum_p a(p)\right]^{1/\mathscr{L}_y} = 1 \tag{21}$$

From Eqs. (13) and (16) we have

$$Z_{\mathscr{L}_y} \sim \Lambda_n^{\mathscr{L}_y} \tag{22}$$

where

$$\Lambda_n = \lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_n} \tag{23}$$

Thus, on the basis of Eqs. (18)–(23) we may write for large $\mathscr{L}_y$

$$Z^{1/|L|} = (\omega_1^{|L|} Z_{\mathscr{L}_y})^{1/|L|} = \omega_1 (\Lambda_n)^{(a_1^2 + a_2^2)/\mathscr{L}_x} \tag{24}$$

where the number of walks n is chosen such that the corresponding eigenvalue is maximal, and the identity $|L| = \mathscr{L}_x \mathscr{L}_y/(a_1^2 + a_2^2)$ has been used.

Now, we have to find values of $k_1, k_2,..., k_n$ which obey (18) and $a(p)$ for which the walk configurations not satisfying the ice rule will cancel out. It is sufficient to write the cancellation conditions of the forbidden configurations at each site of L.

3.1. The Cancellation Conditions

Consider two walks with wave numbers p and q passing through a certain site of L. Let $\mathbb{P}$ and $\mathbb{Q}$ be two permutations of $\{k\}$ in which p and q change the positions: $\mathbb{P}\{k\} = \cdots p, q,...$ and $\mathbb{Q}\{k\} = \cdots q, p,...$. In Fig. 4 four possible configurations of walks corresponding to the first permutation are shown. For another permutation one must change the positions of

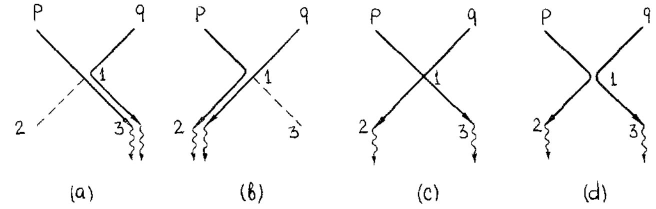

Fig. 4. Four possible walk configurations at a site. Cases (a)–(c) must be eliminated; case (d) is described in the text.

p and q. Numbers 1, 2, 3 denote the sites of L. The segments of walks 1–2 and 1–3 are determined. The wavy lines denote a continuation of the unrestricted random walks.

According to the definitions of Section 2, the weighted sum over all t-step walks from site 1 in the case of Fig. 4a is

$$\frac{\omega_3\omega_5}{\omega_1^2} e^{ipa_2} e^{iqa_2} \tilde{w}_{t-a_1}(p)\, \tilde{w}_{t-a_1}(q) \tag{25}$$

for Fig. 4b is

$$\frac{\omega_4\omega_6}{\omega_1^2} e^{-ipa_1} e^{-iqa_1} \tilde{\tilde{w}}_{t-a_2}(p)\, \tilde{\tilde{w}}_{t-a_2}(q) \tag{26}$$

and for Fig. 4c is

$$\frac{\omega_3\omega_4}{\omega_1^2} e^{ipa_2} e^{-iqa_1} \tilde{\tilde{w}}_{t-a_2}(q)\, \tilde{w}_{t-a_1}(p) \tag{27}$$

The cases of Figs. 4a–4c do not satisfy the ice rule, so they must be elimiated. The case of Fig. 4d corresponds both to a "real collision" satisfying the ice rule and an "imaginary collision" associated with free random walks. So, we have to eliminate the weighted sum

$$\frac{\omega_5\omega_6 - \omega_1\omega_2}{\omega_1^2} e^{-ipa_1} e^{iqa_2} \tilde{\tilde{w}}_{t-a_2}(p)\, \tilde{w}_{t-a_1}(q) \tag{28}$$

Let $A(\mathbb{P})$ and $A(\mathbb{Q})$ be the weight factors of two walks at site 1 multiplied by the coefficients $a(\mathbb{P})$ and $a(\mathbb{Q})$. Using Eqs. (25)–(28) and (15), we can write the cancellation conditions at site 1 as

$$\begin{aligned} A(\mathbb{P})[&\omega_4\omega_6 e^{-ipa_1-iqa_1} + \omega_3\omega_5 e^{ipa_2+iqa_2}\phi(p)\,\phi(q) \\ &+ \omega_3\omega_4 e^{-iqa_1+ipa_2}\phi(p) + (\omega_5\omega_6-\omega_1\omega_2)e^{-ipa_1+iqa_2}\phi(q)] \\ = A(\mathbb{Q})\{&\omega_4\omega_6 e^{-ipa_1-iqa_1} + \omega_3\omega_5 e^{ipa_2+iqa_2}\phi(p)\,\phi(q) \\ &+ \omega_3\omega_4 e^{-iqa_1+iqa_2}\phi(q) + (\omega_5\omega_6-\omega_1\omega_2)e^{-iqa_1+ipa_2}\phi(p)\} \end{aligned} \tag{29}$$

Defining the function $B(p, q)$ by the identity

$$A(\mathbb{P}) = B(p, q)\, A(\mathbb{Q}) \tag{30}$$

we get

$$\begin{aligned} B(p, q) = &-[\omega_4\omega_6 + \omega_3\omega_5 e^{i(p+q)(a_1+a_2)}\phi(p)\,\phi(q) \\ &+ \omega_3\omega_4 e^{iq(a_1+a_2)}\phi(q) + (\omega_5\omega_6-\omega_1\omega_2)e^{ip(a_1+a_2)}\phi(p)] \\ &\times [\omega_4\omega_6 + \omega_3\omega_5 e^{i(p+q)(a_1+a_2)}\phi(p)\,\phi(q) \\ &+ \omega_3\omega_4 e^{ip(a_1+a_2)}\phi(p) + (\omega_5\omega_6-\omega_1\omega_2)e^{iq(a_1+a_2)}\phi(q)]^{-1} \end{aligned} \tag{31}$$

Following the trajectories of walkers from the top row to the bottom one, we note that the forbidden configurations appear as a result of the collision of exactly two walkers at a site. Canceling these configurations subsequently, we satisfy the ice rule for all trajectories on the whole lattice.

For the ice model ($\omega_1 = \omega_2 = \cdots = \omega_6 = 1$), Eq. (31) becomes

$$B(p, q) = -\frac{1 + e^{i(p+q)(a_1+a_2)}(\lambda_p \lambda_q)^{a_2 - a_1} + e^{iq(a_1+a_2)} \lambda_q^{a_2 - a_1}}{1 + e^{i(p+q)(a_1+a_2)}(\lambda_p \lambda_q)^{a_2 - a_1} + e^{ip(a_1+a_2)} \lambda_p^{a_2 - a_1}} \tag{32}$$

where λ_k is the maximal root of the equation

$$\lambda_k^{a_2} e^{ika_1} - \lambda_k^{a_2 - a_1} e^{ik(a_1+a_2)} - 1 = 0 \tag{33}$$

3.2. The Boundary Conditions

Let us consider the weight factors $A(\mathbb{P})$ and $A(\mathbb{Q})$ in more detail. Denote by x_0 and $x_0 + \delta$ the initial positions of two walkers with wave numbers p and q. Let x be the coordinate of a site where the walkers collide. Then the weight factor at the point x for the permutation $\mathbb{P}$ has the form

$$A(\mathbb{P}) = a(\mathbb{P}) e^{ip(x - x_0)} e^{iq(x - x_0 - \delta)} \Omega(\omega_\xi) \tag{34}$$

and for the permutation $\mathbb{Q}$

$$A(\mathbb{Q}) = a(\mathbb{Q}) e^{iq(x - x_0)} e^{ip(x - x_0 - \delta)} \Omega(\omega_\xi) \tag{35}$$

where $\Omega(\omega_\xi)$ is the Boltzmann weight of the walks. Substituting Eqs. (34) and (35) into Eq. (30), we obtain

$$a(\mathbb{P}) = a(\mathbb{Q})\, B(p, q) e^{i\delta(q - p)} \tag{36}$$

Let further $x_1^0, x_2^0, \ldots, x_n^0$ be the initial positions of the walkers with wave numbers $k_1, k_2, \ldots, k_n$ in the permutation $\mathbb{P}$. Denote the difference $x_{j+1}^0 - x_j^0$ by $\delta_{j,j+1}$. The cyclic permutation $\mathbb{Q} = \{2, 3, \ldots, n, 1\}$ gives the coefficient $a(\mathbb{Q})$ associated with $a(\mathbb{P})$ by

$$a(\mathbb{P}) = a(\mathbb{Q}) \prod_{j=2}^{n} B(k_1, k_j) e^{i\delta_{1,2}(k_2 - k_1)} \cdots e^{i\delta_{n-1,n}(k_n - k_1)} \tag{37}$$

On the other hand, the relative positions of the walkers in $\mathbb{P}$ and $\mathbb{Q}$ coincide. So, we must satisfy the identity $a(\mathbb{P}) = a(\mathbb{Q})$. Then, we get

$$e^{i\delta_{1,2}(k_1 - k_2)} \cdots e^{i\delta_{n-1,n}(k_1 - k_n)} = \prod_{j=2}^{n} B(k_1, k_j) \tag{38}$$

Noting that $\sum_{j=1}^{n} \delta_{j,j+1} = \mathscr{L}_x$, one can represent the left-hand side of Eq. (38) as

$$e^{ik_1 \mathscr{L}_x} \mathscr{D}(\{k\}) \tag{39}$$

where

$$\mathscr{D}(\{k\}) = \prod_{j=1}^{n} \exp(-i\delta_{j,j+1} k_j) \tag{40}$$

In principle, one should try to prove that in the thermodynamic limit $\mathscr{L}_x \to \infty$, $n \to \infty$ ($n/\mathscr{L}_x = \text{const}$), $\mathscr{D}(\{k\}) \to 1$ under sufficiently general conditions. However, it is easier to choose all $\delta_{j,j+1}$ to be equal to each other. Then the identity

$$\mathscr{D}(\{k\}) = 1 \tag{41}$$

is fulfilled for any symmetrical set $\{k\}$ in (18). It is not necessary to associate the special choice of $\delta_{j,j+1}$ with the real initial positions of walks, because one can always ascribe an arbitrary initial factor to the weight of a walk. Using Eq. (41), we may rewrite Eq. (38) in the form

$$e^{ik_j \mathscr{L}_x} = \prod_{i \neq j}^{n} B(k_j, k_i) \tag{42}$$

The system of n equations (42) with Eqs. (11) and (31) is a generalization of the Bethe ansatz to the six-vertex model on a rotated square lattice.

4. COMPARISON WITH KNOWN RESULTS

The Bethe ansatz equations for the six-vertex model were obtained in two particular cases: for the "natural" lattice orientation[1] and for the lattice rotated through the angle $\pi/4$.

We begin with the second case. It corresponds to $a_1 = a_2 = 1$. Substituting these values into Eq. (8) and solving Eq. (11), we obtain

$$z_k^{-1} = \lambda_k = \frac{\omega_3}{\omega_1} e^{ik} + \frac{\omega_6}{\omega_1} e^{-ig(k)} \tag{43}$$

where $g(k)$ is defined by the identity

$$\omega_3 e^{ik} + \omega_6 e^{-ig} = \omega_4 e^{-ik} + \omega_5 e^{ig} \tag{44}$$

Using Eqs. (15) and (31) with (43) and (44), we get

$$B(p, q) = -\frac{\omega_4\omega_6 + \omega_3\omega_4 T_q + \omega_3\omega_5 T_p T_q + (\omega_5\omega_6 - \omega_1\omega_2) T_p}{\omega_4\omega_6 + \omega_3\omega_4 T_p + \omega_3\omega_5 T_p T_q + (\omega_5\omega_6 - \omega_1\omega_2) T_q} \tag{45}$$

where $T_k = \exp\{i[k + g(k)]\}$. Denote the number of sites in a row of L by N. Then $\mathscr{L}_x = 2N$, and Eqs. (42) have the form

$$e^{2iNk_j} = \prod_{i \neq j}^{n} B(k_j, k_i) \tag{46}$$

Expressions (44)–(46) coincide with the ones obtained by Pegg.[4]

The natural lattice orientation corresponds to the limit $a_2 \to \infty$, $a_1 = 1$. So, putting $\alpha = ka_2$ and $\lambda_\alpha = \lambda_k^{a_2}$, we get from Eqs. (8), (11), and (15)

$$\lambda_\alpha = \frac{ba - e^{i\alpha}(b^2 - c^2)}{a - be^{i\alpha}} \tag{47}$$

$$\phi_{(\alpha)} = \frac{c}{a - be^{i\alpha}} \tag{48}$$

where $a = \omega_1 = \omega_2$, $b = \omega_3 = \omega_4$, $c = \omega_5 = \omega_6$. Equation (31) becomes

$$B(\alpha, \beta) = -\frac{1 + e^{i(\alpha+\beta)} - 2\Delta e^{i\alpha}}{1 + e^{i)\alpha+\beta)} - 2\Delta e^{i\beta}} \tag{49}$$

with $\alpha = pa_2$, $\beta = qa_2$, $\Delta = (a^2 + b^2 - c^2)/2ab$. Noting that $N = \mathscr{L}_x/a_2$, we get a system of equations for the wave numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$:

$$e^{iN\alpha_j} = -\prod_{i \neq j} B(\alpha_j, \alpha_i) \tag{50}$$

Equations (49) and (50) are the standard Bethe ansatz equations of the six-vertex model.[1] The partition function per row can be obtained from Eqs. (22)–(24):

$$Z^{1/\mathscr{L}_y} = \omega_1^N \prod_{j=1}^{n} \lambda_j \tag{51}$$

This formula differs from the known expression[3] by a term which vanishes in the thermodynamic limit.

5. NUMERICAL RESULTS AND CONCLUSION

Investigation of the obtained equations with an arbitrary angle of rotation requires numerical calculations.

We shall consider equations for the ice model on the rotated lattice given by Eqs. (32), (33), and (42). The simplest case which is far from the known results is $a_1 = 1$, $a_2 = 2$. In this case Eq. (33) gives

$$\lambda_k = \frac{e^{2ik}}{2}(1 + S_k) \tag{52}$$

where $S_k = (1 + 4e^{-5ik})^{1/2}$. Equation (32) becomes

$$B(k, k') = -\frac{3 - S_{k'} + S_k + S_k S_{k'}}{3 - S_k + S_{k'} + S_k S_{k'}} \tag{53}$$

As usual, for large n, Eq. (42) becomes an integral equation for the wave-number density $\rho(k)$

$$2\pi\rho(k) = 1 + \int_{-Q}^{Q} \frac{\partial\theta(k, k')}{\partial k} \rho(k')\, dk' \tag{54}$$

where the function $\theta(k, k')$ is defined by

$$B(k, k') = -e^{-i\theta(k,k')} \tag{55}$$

and an interval $(-Q, Q)$ is defined by the normalization condition

$$\int_{-Q}^{Q} \rho(k)\, dk = \frac{n}{\mathscr{L}_x} \tag{56}$$

One can deduce from Eqs. (53) and (55) that

$$\operatorname{Im} \frac{\partial\theta(k, -k')}{\partial k} = -\operatorname{Im} \frac{\partial\theta(k, k')}{\partial k} \tag{57}$$

and

$$\operatorname{Re} \frac{\partial\theta(k, -k')}{\partial k} = \operatorname{Re} \frac{\partial\theta(k, k')}{\partial k} \tag{58}$$

Then Eq. (54) becomes

$$2\pi\rho(k) = 1 + \operatorname{Re} \int_{-Q}^{Q} \frac{\partial\theta(k, k')}{\partial k} \rho(k')\, dk' \tag{59}$$

and gives a real, symmetric, positive solution $\rho(k)$. In the case of $a_1 = 1$, $a_2 = 2$ we have to choose $n/\mathscr{L}_x = 3/10$, which corresponds to the density of walks providing the maximum of entropy. Figure 5 displays the spectrum

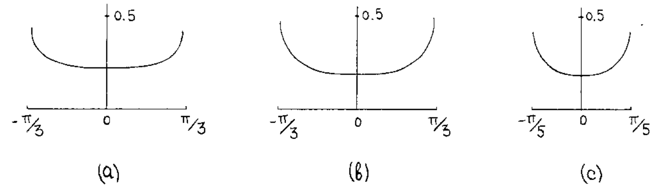

Fig. 5. The density function $\rho(k)$ for the cases of (a) $a_1 = 1$, $a_2 = \infty$; (b) $a_1 = a_2 = 1$; (c) $a_1 = 1$, $a_2 = 2$.

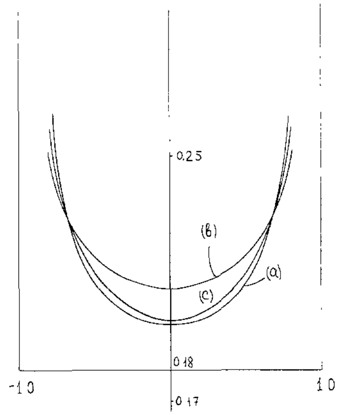

Fig. 6. The curves of Fig. 5 in units of the Fermi momentum.

of k for the cases (a) $a_1 = 1$, $a_2 = \infty$; (b) $a_1 = a_2 = 1$; and (c) $a_1 = 1$, $a_2 = 2$. In Fig. 6 we plot the same curves in units of the Fermi momentum. We see that the curves differ little from each other, and curve (c) lies between the limiting curves (a) and (b). As expected, all three cases of the lattice orientation give the same value of the entropy per site $S = \frac{3}{2} \ln \frac{4}{3}$.[1]

In conclusion, we have obtained the Bethe ansatz equations for the six-vertex model on the lattice rotated through an arbitrary angle. In terms of random walks, these equations represent cancellation conditions of the forbidden trajectories. The visualizability and relative simplicity of this method make us hope that the six-vertex model can be solved on an open strip, and further information can thus be obtained about thermodynamic properties of this model.

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